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High Accuracy Iterative Solution of Convection Diffusion Equation with Boundary Layers on Nonuniform Grids[1]

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A fourth-order compact finite difference scheme and a multigrid method are employed to solve the two-dimensional convection diffusion equations with boundary layers. The computational domain is first discretized on a nonuniform (stretched) grid to resolve the boundary layers. A grid transformation technique is used to map the nonuniform grid to a uniform one. The fourth-order compact scheme is applied to the transformed uniform grid. A multigrid method is used to solve the resulting linear system. Numerical experiments are used to show that a graded mesh and a grid transformation are necessary to compute high accuracy solutions for the convection diffusion problems with boundary layers and dicretized by the fourth-order compact scheme.

Key Words: convection diffusion equation; boundary layer; grid stretching; grid transformation; multigrid method.

1. INTRODUCTION

Numerical simulation of the convection diffusion equation plays a very important role in computational fluid dynamics to simulate flow problems. A two-dimensional convection diffusion equation satisfying Dirichlet boundary conditions can be written in the form of

$$\begin{aligned} u_{xx} + u_{yy} + p(x,y)u_x + q(x,y)u_y &= f(x,y), \quad (x,y) \in \Omega, \\ u(x,y) &= g(x,y), \quad (x,y) \in \partial\Omega. \end{aligned} \tag{1}$$

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The convection coefficients $p(x, y)$ and $q(x, y)$ are functions of the independent variables x and y, and are assumed to be sufficiently smooth. Here Ω is a convex domain consisting of a union of rectangles, and $\partial\Omega$ is the boundary of Ω. The magnitude of $p(x, y)$ and $q(x, y)$ may be referred to as the Reynolds number (Re), and it determines the ratio of the convection to diffusion. Numerical solutions of Eq. (1) based on iterative solution methods become increasingly difficult (converge slowly or even diverge) as the ratio of the convection to diffusion increases [35]. Traditional finite difference discretization schemes such as the second-order central difference scheme and the first-order upwind scheme have the drawbacks of either lack of stability (central difference) or lack of accuracy (upwind). There is considerable interest in developing improved finite difference discretization schemes for the convection diffusion equations [1, 11, 13, 14]. Recently, the class of higher order compact discretization schemes with superconvergent properties has attracted much attention and has been applied to the convection diffusion equations [5, 8, 9, 12, 23, 31].

In the various ways of differencing Eq. (1), the most familiar schemes are the central difference scheme and the upwind difference scheme. These two schemes yield a linear system with a five-point sparse matrix of the form

$$Au = f. \tag{2}$$

In the case of the central difference scheme, classical iterative methods for solving the resulting linear system (2) do not converge when the convective terms dominate and when the cell Reynolds number is greater than a certain constant. Conventional upwind difference approximation is computationally stable, but is only first-order accurate; and the resulting solution exhibits the effect of artificial viscosity [18, 21].

For the two-dimensional convection diffusion equation, Gupta *et al.* [8] proposed a fourth-order nine-point compact finite difference formula, which was shown to be computationally efficient and stable, and to yield highly accurate numerical solutions. The resulting linear system can be solved by classical iterative methods for large values of the Reynolds number [8]. Zhang [27, 29] and others [7] proposed a few multigrid methods with acceleration schemes and special intergrid transfer operators to solve the linear systems arising from the fourth-order compact discretization of Eq. (1) with high Reynolds numbers.

Although considerable amount of work has been done in the past, there is still a lack of a completely satisfactory computational (discretization and solution) scheme that is suitable for all types of convection diffusion equations [35]. Most applications of practical interest require some form of mesh grading or local mesh refinement to deal with locally rapidly changing solutions [6, 21]. The problem of constructing high-order compact schemes on nonuniform grids has been raised recently, and a few diffusion dominated boundary layer problems have been solved to show the effect of using nonuniform grids with the fourth-order compact scheme [9, 24]. The present work is to employ the ideas of [9, 24] to solve boundary layer problems with high Reynolds numbers, and to investigate the convergence behavior of multigrid method for solving the resulting linear system (with grid stretching).

In this paper, we solve Eq. (1) with boundary layers and with Reynolds numbers up to 10^7 using a fourth-order compact scheme on a graded mesh with a coordinate transformation technique and a multigrid method. We discuss the fourth-order compact discretization scheme for transformed convection diffusion equation in Section 2. In Section 3, we briefly introduce the multigrid method and outline its advantages to solve the relevant sparse linear systems. Numerical experiments are conducted in Section 4 to show the necessity of the graded mesh and a grid transformation for computing high-accuracy solutions. Some concluding remarks are summarized in Section 5.

2. FOURTH-ORDER COMPACT DIFFERENCE SCHEME

Fourth-order nine-point compact finite difference schemes for Eq. (1) on a rectangular uniform grid have been designed by several authors [5, 8, 12, 22, 23]. It is believed that these schemes are mathematically equivalent, although they were derived from different approaches. We mainly focus on the particular formula given by Gupta *et al.* [8], since we will use their technique on a nonuniform grid in the present work [9]. Let Ω be discretized on a uniform grid in both x and y dimensions. The approximate value of a function $u(x, y)$ at a reference mesh point (x, y) is denoted by u_0. The approximate values at its eight immediate neighboring mesh points are denoted by $u_i,\ i = 1, 2, \ldots, 8$. The compact nine grid points are labeled as follows

$$\begin{pmatrix} u_6 & u_2 & u_5 \\ u_3 & u_0 & u_1 \\ u_7 & u_4 & u_8 \end{pmatrix}.$$

The compact finite difference formula for the mesh point (x, y) involves the nearest eight neighboring mesh points with the uniform mesh size h. Since we will not use this fourth-order compact scheme directly on Eq. (1) with boundary layers, for the sake of saving space, we refer readers to the original paper for the details of the formula and the derivation procedure [8]. This discretization scheme has been shown to be computationally efficient and numerically stable with respect to the application of iterative techniques, in addition to producing high-accuracy approximate solutions for smooth functions [9, 29, 32]. The unconditional stability of the fourth-order compact scheme makes it very attractive in use with multigrid method [29]. Since the iterations on all coarse grids converge, they provide sufficiently accurate corrections to the fine-grid iteration. In other words, there is no cell Reynolds number effect on the coarse grids [35]. This is an advantage that is not shared by the central difference scheme.

2.1. *Transformed Convection Diffusion Equation*

For many convection diffusion problems encountered in practical applications, the computational domain usually contains steep boundary layers in which the solution fluctuates rapidly. For such problems, the numerical solution from the central difference scheme may exhibit nonphysical oscillations if the mesh size is not fine enough. Although such oscillations can be suppressed by the use of the upwind difference scheme, the accuracy of the computed solution from this scheme is reduced to the first order. Very fine discretization has to be used to yield an approximate solution of acceptable accuracy. Such a fine discretization results in very large linear systems that demand large computational effort.

It has been shown that the fourth-order compact scheme can suppress the nonphysical oscillations to a certain degree [35]. For one-dimensional model problems, it can be shown that the computed solution with the fourth-order compact scheme is nonoscillatory [22]. However, numerical and analytic studies indicate that the order of the computed solution from the fourth-order compact scheme may be reduced to $O(h^2)$ when the Reynolds number is large [21, 26, 35]. For probems with boundary layers, the accuracy of the computed solutions from the three difference schemes mentioned in this paper is undesirable [35]. The advantages of the fourth-order compact scheme may be lost if there are no mesh points inside the boundary layers. To obtain a high accuracy solution to the boundary layer problems, suitable techniques must be utilized to place a certain number of mesh points inside the boundary layers. To avoid too many grid points in the computational domain and to reduce the total computational cost, the smooth region of the domain should be placed with relatively few grid points. This leads to the requirement of graded mesh techniques, local mesh refinement strategies, or grid adaptive algorithms [6, 16, 21]. However, the existing fourth-order compact scheme and the high-order compact methodology can only work on a uniform grid.[3] A typical solution to such a conflict is to use a coordinate transformation technique to map a graded mesh to a uniform mesh, so that the fourth-order compact scheme can be applied on the transformed uniform grid [3]. This is the approach that will be used in our study. Similar approaches have been used by Gupta *et al.* [9] and by Spotz and Carey [24].

Consider a nondegenerate map $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, which transforms Eq. (1) from a graded mesh on $0 < x < 1, 0 < y < 1$ to a uniform mesh on $0 < \xi < 1, 0 < \eta < 1$. The transformed equation can be written as

$$\alpha(\xi, \eta)u_{\xi\xi} + \beta(\xi, \eta)u_{\eta\eta} + c(\xi, \eta)u_{\xi\eta} + \lambda(\xi, \eta)u_\xi + \mu(\xi, \eta)u_\eta = f(\xi, \eta), \tag{3}$$

where the coefficients are given by

$$\begin{aligned}
\alpha(\xi, \eta) &= \xi_x^2 + \xi_y^2, \quad \beta(\xi, \eta) = \eta_x^2 + \eta_y^2, \\
\lambda(\xi, \eta) &= p(\xi, \eta)\xi_x + q(\xi, \eta)\xi_y + \xi_{xx} + \xi_{yy}, \\
\mu(\xi, \eta) &= p(\xi, \eta)\eta_x + q(\xi, \eta)\eta_y + \eta_{xx} + \eta_{yy}, \\
c(\xi, \eta) &= 2(\xi_x\eta_x + \xi_y\eta_y).
\end{aligned}$$

The difference between the transformed Eq. (3) and the original Eq. (1) is the variable diffusion coefficients α, β, and c of the second-order derivative terms appearing in the transformed equation, but not in the original equation. In particular, the second-order cross derivative term $u_{\xi\eta}$ in Eq. (3) may present some problems to the fourth-order compact formulation [24]. Fortunately, for the orthogonal grids used in our current study, the coefficient $c(\xi, \eta)$ is identically zero throughout Ω. Hence, Eq. (3) is simplified as

$$\alpha(\xi, \eta)u_{\xi\xi} + \beta(\xi, \eta)u_{\eta\eta} + \lambda(\xi, \eta)u_\xi + \mu(\xi, \eta)u_\eta = f(\xi, \eta). \tag{4}$$

2.2. *Fourth-Order Approximation for Transformed Equation*

To derive a fourth-order compact finite difference approximation, we assume that the solution $u(\xi, \eta)$ of Eq. (4) satisfies the Taylor series expansion

$$u(\xi, \eta) = \sum a_{ij}\xi^i\eta^j$$

[3] There are some implicit high-order compact schemes that can work on nonuniform grids [19]. These high-order compact schemes are implicit because they involve discrete values of the first- and second-order derivatives, in addition to the discrete function values.

locally on the mesh points with a local center at the reference grid point 0 and the nearest 8 neighboring grid points. The coefficients α, β, λ, μ, and the forcing function f are assumed to have similar Taylor series expansions on the grid points in question. The fourth-order compact difference scheme can be obtained by substituting the Taylor series expansions into Eq. (4) and by obtaining the representation for u in Eq. (4) to obtain a finite difference formula of order 4. This is achieved by truncating the Taylor series up to order 4 (by setting all the Taylor series coefficients of u_{ij} to zero for $i + j > 4$). The derivation procedure is straightforward but tedious. We omit the details and refer interested readers to the original paper of Gupta *et al.* [9]. We mention that a different procedure was proposed by Spotz and Carey in which the property of the mapping function is also considered [24].

The nine-point compact finite difference approximation of Eq. (4) yields at each internal grid point a linear equation of the form [9]

$$\sum_{j=0}^{8} \alpha_j u_j = 6h^2 f_{00} + h^4[f_{20} + f_{02} + T_1 f_{10} + T_2 f_{01}], \tag{5}$$

where the coefficients are given by

$$\begin{aligned}
&\alpha_0 = 2R_1 + 2R_2 + 4S_1, \quad \alpha_1 = R_1 + R_3,\\
&\alpha_2 = R_2 + R_4, \quad \alpha_3 = R_1 - R_3, \quad \alpha_4 = R_2 - R_4,\\
&\alpha_5 = S_1 + S_2 + S_3 + S_4, \quad \alpha_6 = S_1 + S_2 - S_3 - S_4,\\
&\alpha_7 = S_1 - S_2 + S_3 - S_4, \quad \alpha_8 = S_1 - S_2 - S_3 + S_4,\\
&T_1 = (\lambda_{00} - 2\alpha_{10})/(2\alpha_{00}), \quad T_2 = (\mu_{00} - 2\beta_{01})/(2\beta_{00}),\\
&R_1 = 5\alpha_{00} - \beta_{00} + T_1 h^2(\lambda_{00} + \alpha_{10}) + T_2 h^2 \alpha_{01} + h^2(\alpha_{20} + \alpha_{02} + \lambda_{10}),\\
&R_2 = 5\beta_{00} - \alpha_{00} + T_2 h^2(\mu_{00} + \beta_{01}) + T_1 h^2 \beta_{10} + h^2(\beta_{20} + \beta_{02} + \mu_{01}),\\
&R_3 = h/2(5\lambda_{00} - 2\beta_{10} - 2\beta_{00} T_1) + h^3/2[T_1\lambda_{10} + T_2\lambda_{01} + \lambda_{20} + \lambda_{02}],\\
&R_4 = h/2(5\mu_{00} - 2\alpha_{10} - 2\alpha_{00} T_2) + h^3/2[T_2\mu_{01} + T_1\mu_{10} + \mu_{20} + \mu_{02}],\\
&S_1 = 1/2(\alpha_{00} + \beta_{00}), \quad S_2 = h/4[\mu_{00} + 2\alpha_{01} + 2T_2\alpha_{00}],\\
&S_3 = h^2/4[\mu_{10} + \lambda_{01} + T_1\mu_{00} + T_2\lambda_{00}], \quad S_4 = h/4[\lambda_{00} + 2\beta_{10} + 2T_1\beta_{00}].
\end{aligned}$$

The truncation error of the approximation scheme (5) is $O(h^4)$; see [9] for details. Equation (5) utilizes partial derivatives of the functions α, β, λ, μ, and f. A double subscript "ij" on any of these functions denotes the $(i + j)$th partial derivative defined by

$$\alpha_{ij} = \frac{1}{i!j!} \frac{\partial^{i+j}\alpha}{\partial\xi^i \partial\eta^j}.$$

In this formulation, we require that the partial derivatives in question exist analytically. It is also possible to approximate these partial derivatives by finite difference formulas. However, as Spotz and Carey showed in [24], the finite difference approximations for the partial derivatives must be of fourth-order accuracy in order for the entire approximation scheme to maintain its fourth-order accuracy.

3. MULTIGRID METHOD

Multigrid method is among the fastest and most efficient algorithms for solving linear systems arising from discretized elliptic partial differential equations [2, 25]. The multigrid algorithm iterates on a hierarchy of successively coarser grids until the convergence is reached (the residual equations are approximately solved on the coarse grids); considerable computational time is saved by doing major computational work on the coarse grids. For more details on the motivation, philosophy, and processes of multigrid method, readers are referred to [2, 25] and the references therein. For the specific multigrid methods used to solve the convection diffusion equation with a fourth-order compact discretization scheme, see [7, 10, 15, 27, 29, 32]. In general, the standard multigrid method works well with the fourth-order compact scheme for $\mathrm{Re} \leq 10^3$. For convection dominated problems, acceleration schemes, such as the minimal residual smoothing technique [28], and special intergrid transfer operators, have to be employed to achieve reasonable convergence rates.

For convection diffusion equations discretized by the upwind type schemes, some forms of algebraic multigrid approaches have been shown to be efficient [4, 17, 20]. An ILU preconditioning technique has also been used to solve the sparse linear systems arising from discretized convection diffusion equations with the central difference, upwind difference, and the fourth-order compact schemes [35].

In the present work, we use a standard multigrid approach with an alternating line Gauss–Seidel relaxation [25]. Smoothing analyses in [32] show that the alternating line Gauss–Seidel relaxation is a robust smoother for the convection diffusion equation discretized by the fourth-order compact scheme, although the point Gauss–Seidel relaxation also has smoothing effect for all Reynolds numbers. One presmoothing and one postsmoothing sweeps are performed on each level in a V-cycle algorithm. In addition, a standard bilinear interpolation operator and a full weighting restriction operator are used as the intergrid transfer operators [25].

4. NUMERICAL RESULTS

Two test problems were solved using the discretization and solution techniques discussed in previous sections. The first two subsections report the accuracy of the computed solutions of the fourth-order compact scheme with respect to the grid stretching, assuming that the linear systems were solved successfully by the multigrid method. The third subsection discusses the effect of grid stretching on the convergence of the multigrid method.

4.1. *Problem 1*

We first consider a constant coefficient convection diffusion equation

$$-\epsilon(u_{xx} + u_{yy}) + u_x = 0 \tag{6}$$

defined on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. The boundary condition is prescribed as

$$u(x, 0) = u(x, 1) = 0; \quad u(0, y) = \sin \pi y; \quad u(1, y) = 2 \sin \pi y.$$

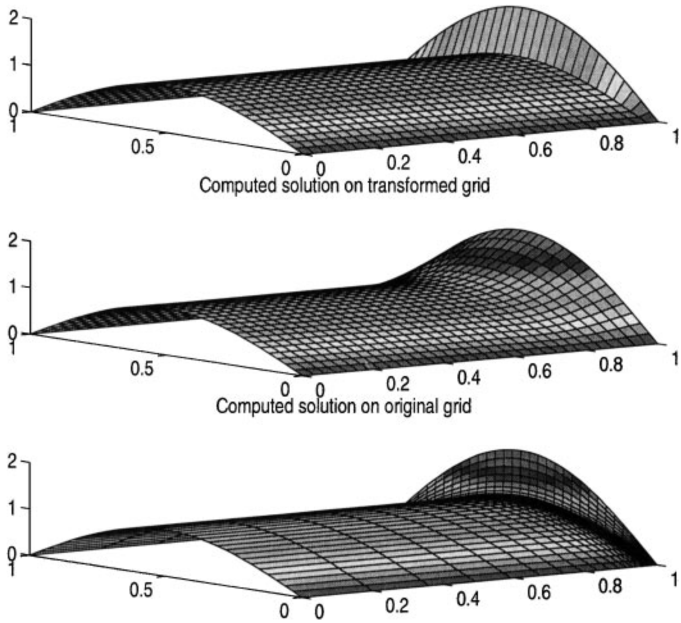

FIG. 1. An illustration of the exact solution on original grid with the uniform mesh, the computed solution on the transformed grid, and the computed solution on the stretched grid for Problem 1 with $\epsilon = 0.001$.

The exact solution is [9]

$$u(x, y) = \exp(x/2\epsilon) \sin \pi y[2 \exp(-1/2\epsilon) \sinh \sigma x + \sinh \sigma (1 - x)]/\sinh \sigma,$$

where $\sigma^2 = \pi^2 + 0.25/\epsilon^2$.

This problem represents a convection dominated flow and was used as one of the test problems by Gupta *et al*. [9], who tested it with ϵ as small as 0.01. The coefficient of the convective term is a constant. The top picture of Fig. 1 is the exact solution of Eq. (6) with $\epsilon = 0.001$ and is shown on a uniform mesh in the original coordinate system. For most part of the domain, the exact solution exhibits smooth values. But it has a steep boundary layer of thickness $O(\epsilon)$ along the downstream edge at $x = 1$, and has shear layers of thickness $O(\sqrt{\epsilon})$ along the top and bottom edges at $y = 0$ and $y = 1$.

Following Gupta *et al*. [9], we used the following coordinate transformation to resolve the boundary layer

$$x = [1 - \exp(-Q\xi)]/[1 - \exp(-Q)]. \tag{7}$$

This transformation maps the interval $0 < x < 1$ onto $0 < \xi < 1$; the y coordinate direction is not changed. A uniform mesh in ξ translates into a graded mesh in x which is coarser near $x = 0$ and finer near $x = 1$ (see the bottom picture of Fig. 1). The parameter Q relates the coarsest mesh width D near $x = 0$ with the finest mesh width d near $x = 1$. The total number of mesh points along the ξ coordinate direction is $Q = \ln(\gamma)/(1 - \Delta\xi)$, where $\gamma = D/d$ is the mesh stretching ratio, and $\Delta\xi = 1/N$ (N is the total number of mesh intervals in the ξ coordinate direction). Using the coordinate transformation of Eq. (7), the partial differential equation (6) becomes

$$\left(\frac{z}{Q}\right)^2 u_{\xi\xi} + u_{yy} - \frac{z}{Q}\left(\frac{1}{\epsilon} - z\right)u_\xi = 0, \tag{8}$$

where $z = \exp(Q\xi)[1 - \exp(-Q)]$. Note that the original Eq. (6) is a constant problem. But the transformed Eq. (8) is a variable coefficient problem; since the parameter z is now a function of the independent variable ξ.

The fourth-order compact scheme discussed in Section 2 is applied to the transformed Eq. (8) on a uniform grid of the ξ and y variables with a uniform mesh size h. The total number of grid points in each coordinate dimension is $N = 1/h + 1$, of which $(N - 1)$ are internal points with unknown values. At each internal grid point, we have a nine-point linear equation of the form (5) and all of the $(N-1)^2$ equations constitute a sparse linear system of the form (2). This sparse linear system is then solved by a multigrid method. For the standard (geometric) multigrid method, we actually discretized Eq. (8) on all coarse meshes (on transformed grids) with a reduction ratio of 2 between the successive meshes, until the coarsest mesh has only one internal grid point. (This is the standard coarsening technique used in geometric multigrid method [25]).

We computed the numerical solutions of Eq. (8) for several values of the perturbation parameter ϵ and the mesh stretching parameter γ. Sample results for $10^{-7} \le \epsilon \le 1$ are given in Table I. The errors reported are the maximum absolute error over all of the discrete grid points. In Table I, the parameter γ controls the amount of grid stretching; $\gamma = 1$ implies no stretching was used. The last column shows the order of accuracy of the discretization scheme and the data were computed by order $= \log_2(\text{error}_{h=1/64}/\text{error}_{h=1/128})$.

TABLE I

Maximum Error in the Computed Solution of Problem 1 with Different Diffusion Coefficient ϵ, Stretching Ratio γ, and Discretization Parameter h

ϵ	$\gamma \backslash h$	1/64	1/128	Order
10^0	1	3.69 (−9)	2.30 (−10)	4.00
	5	1.36 (−7)	8.31 (−9)	4.03
	1	9.01 (−7)	5.62 (−8)	4.00
10^{-1}	10	2.50 (−7)	1.53 (−8)	4.04
	20	2.70 (−7)	1.67 (−8)	4.03
	100	7.70 (−7)	4.75 (−8)	4.02
	1	3.44 (−3)	2.12 (−4)	4.02
	10	3.31 (−5)	2.12 (−6)	3.96
	100	2.03 (−6)	1.27 (−7)	4.00
	1	4.68 (−1)	2.20 (−1)	1.09
10^{-2}	100	3.18 (−4)	2.13 (−5)	3.90
	400	2.20 (−5)	1.43 (−6)	3.96
10^{-3}	4×10^2	2.19 (−5)	1.43 (−6)	3.94
10^{-4}	10^4	3.58 (−5)	2.09 (−6)	4.10
10^{-5}	10^5	9.20 (−5)	5.70 (−6)	4.01
10^{-6}	10^6	1.91 (−4)	1.20 (−5)	3.99
10^{-7}	9×10^7	6.78 (−4)	2.36 (−5)	4.84

It can be seen that the fourth-order compact scheme indeed yields solution of $O(h^4)$ when Eq. (6) does not have a steep boundary layer. For $\epsilon = 0.001$, even the fine mesh size $h = d = 1/128$ without stretching ($\gamma = 1$) could not resolve the boundary layer. The boundary layer has a thickness of 0.001, and the mesh size $d = 1/128 = 0.0078$ implies that there was no grid point placed inside the boundary layer. The computed solution from the fourth-order compact scheme has an accuracy order of 1.09 (see Table I, row 10 and column 5). However, when the mesh was stretched, the fourth-order accuracy is recovered with $\gamma = 100$. Within a certain range of grid stretching, the larger the stretching ratio γ, the more accurate the computed solution. However, too much stretching may have a negative effect. This is demonstrated by the results in rows 5 and 6 with $\epsilon = 0.1$; the use of the stretching ratios $\gamma = 20$ and $\gamma = 100$ yielded computed solution with a lower accuracy than that with a stretching ratio $\gamma = 10$. This is because, in these cases, the boundary layer is not too steep. A larger stretching ratio puts too many grid points along the boundary $x = 1$ and too few along the boundary $x = 0$. Thus, the maximum absolute error was actually found close to the boundary $x = 0$, not close to the boundary layer at $x = 1$, as one would expect.

From Table I, we can see that, with a nonuniform grid, a boundary layer problem with large values of Re (small ϵ) (as large as 10^7) can be solved. This table indicates that the steeper the boundary layer, the larger the stretching ratio is needed. Most importantly, it shows that the fourth-order accuracy rate is achieved even when the problem is highly convection dominated. Simply put, the high-order accuracy of the fourth-order compact scheme is recovered by the use of grid stretching strategy. This result is surprising since several authors have shown that the fourth-order compact scheme yields computed solution of second-order accuracy at high Reynolds numbers on uniform grids [21, 26, 35]. Our numerical results show that the fourth-order compact scheme may still maintain its high accuracy at high Reynolds numbers on nonuniform grids.

We also show in Fig. 1 the exact solution ($\epsilon = 0.001$, $h = 1/32$) on the original grid with a uniform mesh, the computed solution on the transformed grid, and the computed solution on the original grid with a stretched mesh. We can see that there is no grid point inside the boundary layer on the original uniform grid. The middle picture of Fig. 1 gives an impression that the transformation of the stretched grid to a uniform grid has the effect of smothing the boundary layer so that a few grid points can be placed into the region near $\xi = 1$. In fact, the boundary layer almost disappeared on the transformed grid. It seems that the solution on the stretched grid is smooth and the fourth-order compact approximation is applicable. The bottom picture of Fig. 1 shows that with a mesh stretching ratio $\gamma = 10^3$, the grid points are clustered along the boundary layer at $\xi = 1$. There are a few grid points placed inside the boundary layer, and the computed solution is shown to be in good agreement with the exact solution.

Figure 2 plots the solution profiles at $\xi = 1 - h$ when a different mesh size was used; the other parameters were set as $\epsilon = 0.001$, $\gamma = 5$. The solution accuracy is rapidly improved with the increase in the number of grid points. Comparison of the computed solutions with different stretching ratios is given in Fig. 3, where the parameters were set as $\epsilon = 0.001$, $h = 1/64$, and the stretching ratios were set at 1, 5, 10, and 100, respectively.[4] Once again, the more severe the grid stretching, the more accurate the computed solution. When

[4] The solutions shown in four subfigures of Fig. 2 and in other similar figures are not necessarily the same solution. They are the solutions relative to the particular values of h and γ.

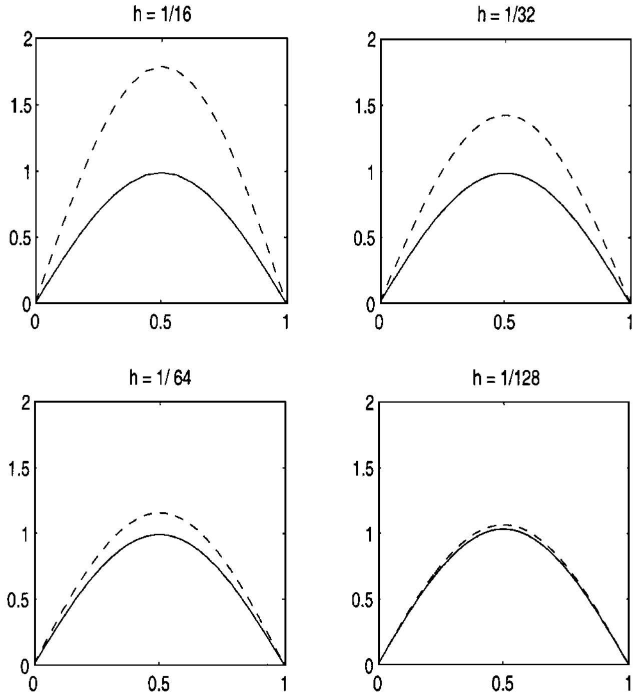

FIG. 2. Comparison of the exact solution and the computed solution of Problem 1 ($\epsilon = 0.001$) with a stretching ratio $\gamma = 5$ at $\xi = 1 - h$ with different mesh size. Solid line is the exact solution, dashed line is the computed solution.

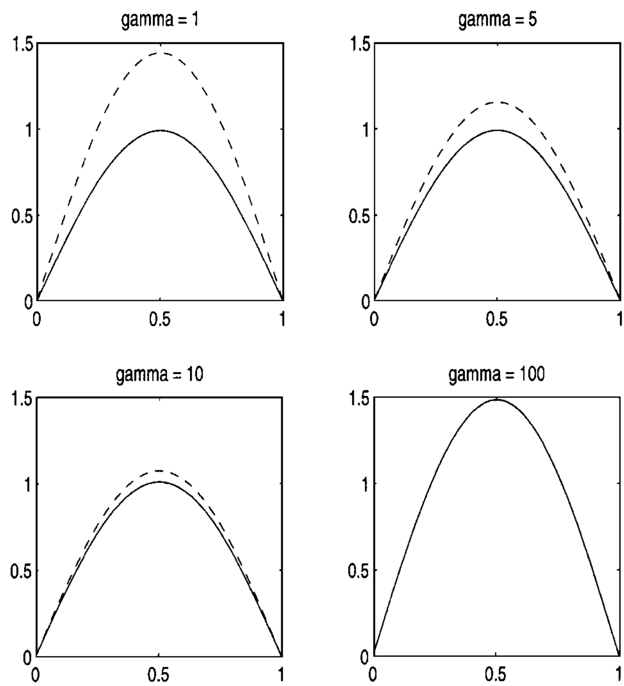

FIG. 3. Solution profiles of Problem 1 ($\epsilon = 0.001$) with different stretching ratios at $\xi = 1 - h$ with $h = 1/64$. Solid line is the exact solution, dashed line is the computed solution.

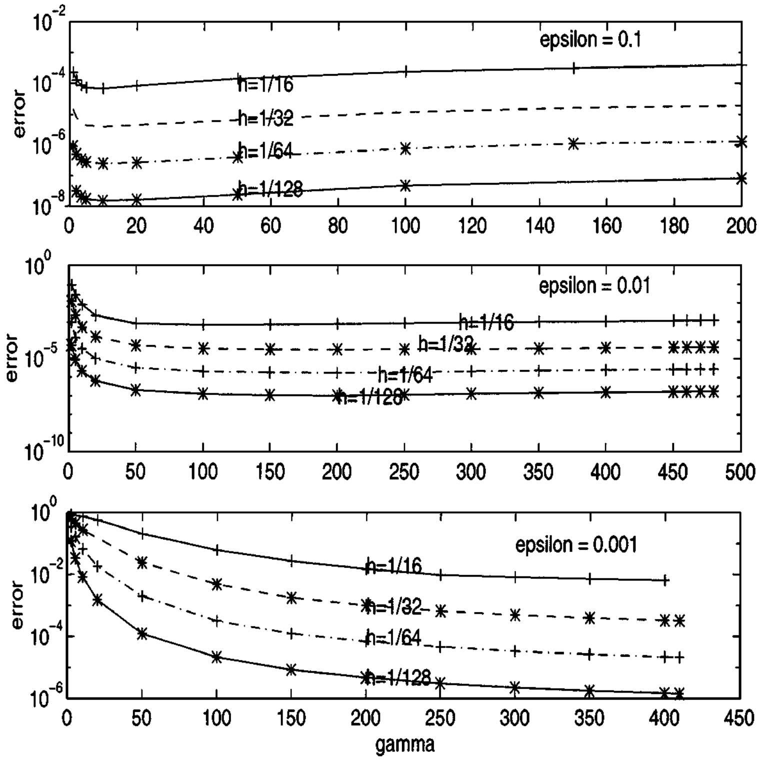

FIG. 4. Maximum absolute errors of the computed solution of Problem 1 with different stretching ratios γ and different mesh size h.

$\gamma = 100$, we had the most accurate solution. In this case, the computed solution is graphically indistinguishable with the exact solution.

Figure 4 shows how the location of the maximum absolute errors changes with different stretching ratios and with different mesh size. For $\epsilon = 0.1$, we obtained the highest accuracy at $\gamma = 5$. When the stretching ratio increased, the maximum absolute error increased because of the dominance of the errors at near $x = 0$, which is in agreement with the results of Table I. For a medium thickness boundary layer with $\epsilon = 0.01$, the maximum absolute errors keep as a constant. For a much steeper boundary layer, e.g., for $\epsilon = 0.001$, the maximum absolute errors keep as a constant after a much larger stretching ratio. Figure 5 is the error distribution contours in the computational domain. It gives results similar to those in Table I and in Fig. 4. It is clear that a different stretching ratio γ results in the maximum absolute error being found at different locations.

4.2. *Problem 2*

For the second test problem, we chose in Eq. (1) the following convection coefficients

$$p(x, y) = \text{Re}\, x(x - 1)(1 - 2y), \quad q(x, y) = -\text{Re}\, y(y - 1)(1 - 2x).$$

It is obvious that this test problem has a stagnation point at (0.5, 0.5). A stagnation point (x_0, y_0) is the one inside the computational domain where both convection coefficients vanish; i.e., $|p(x_0, y_0)| + |q(x_0, y_0)| = 0$. Convection diffusion equations with stagnation points in their domains are usually used to model recirculation flow problems. For large Re, this type of problem is very hard to solve, especially when standard multigrid method is used as the solution technique [29]. The exact solution is chosen similar to the one used by Spotz and Carey [24] as

$$u(x, y) = [1 - b(x)][1 - b(y)],$$

where

$$b(x) = [\exp(-\mathrm{Re}(x-1)) - 1]/[\exp(\mathrm{Re}) - 1],$$
$$b(y) = [\exp(-\mathrm{Re}(y-1)) - 1]/[\exp(\mathrm{Re}) - 1].$$

The exact solution has steep boundary layers near $x = 0$ and $y = 0$ (see the top picture of Fig. 6. To solve this problem with the boundary layer, we introduce the mapping functions [24]

$$x(\xi) = \xi + \frac{\gamma}{\pi}\sin\pi\xi, \quad y(\eta) = \eta + \frac{\gamma}{\pi}\sin\pi\eta,$$

where γ is the grading parameter. The mapping function is invertible for $|\gamma| < 1$. The expression $\gamma > 0$ corresponds to a compression (clustering) to the right ($x = 1$) and similarly to the left ($x = 0$) for $\gamma < 0$. In the present calculation, we select $\gamma = 0.0, -0.2, -0.5,$

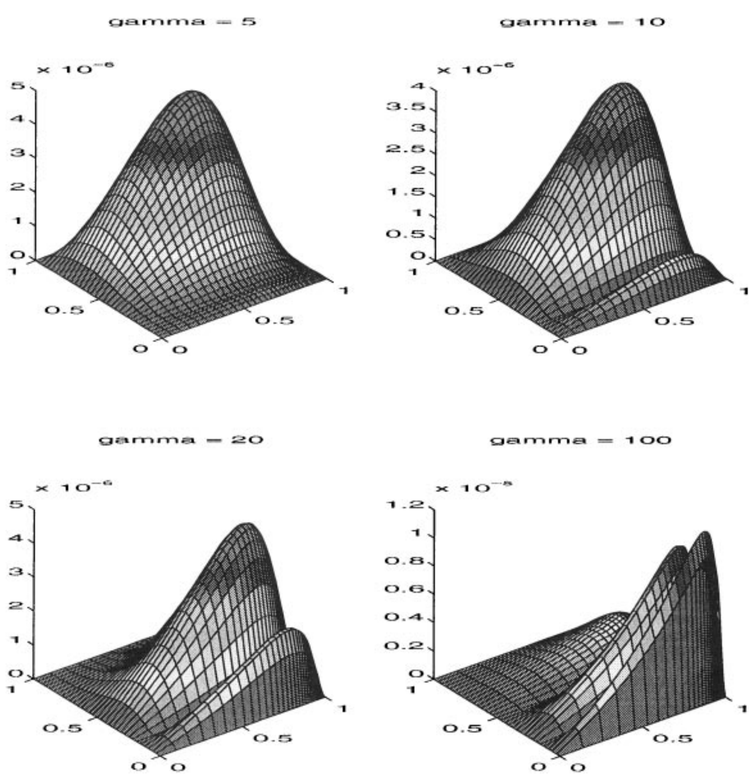

FIG. 5. Absolute error distribution contours of Problem 1 ($\epsilon = 0.01$) with different stretching ratios and $h = 1/32$.

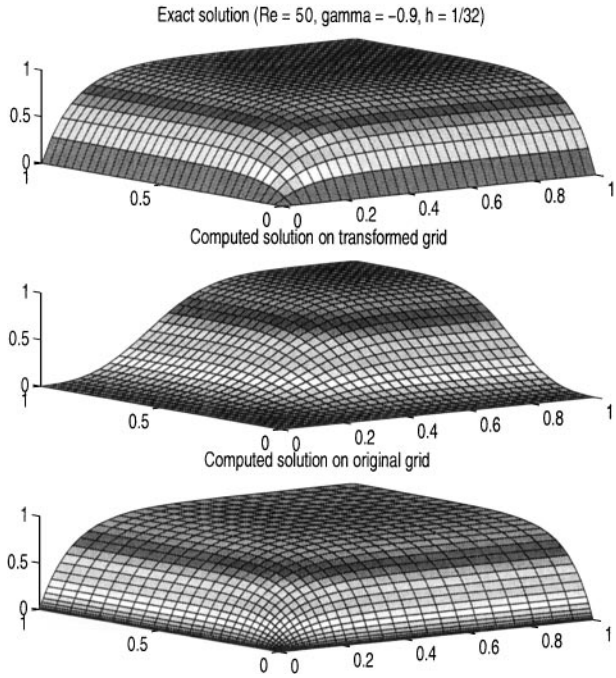

FIG. 6. An illustration of the exact solution on the original grid with the uniform mesh, the computed solution on the transformed grid, and the computed solution on the original stretched grid of Problem 2 with Re $= 50$, $\gamma = -0.9$, $h = 1/32$.

-0.7, -0.9 to compare the effect of the grid stretching ratio on the accuracy of the computed solution. The test problem was computed on a sequence of uniformly refined grids with $h = 1/16$, $1/32$, $1/64$, $1/128$ for several Reynolds numbers Re $= 1$, 50, 100, 250, 500, 1000. Figure 6 shows the exact solution on the original uniform grid, the computed solution on the transformed grid, and the computed solution on the original stretched grid, respectively. The parameters were chosen as Re $= 50$, $\gamma = -0.9$, and $h = 1/32$. Because of the high stretching ratio, we obtain a very accurate computed solution (see the bottom picture of Fig. 6). The comparison of the maximum absolute errors with different mesh size and several stretching ratios ($\gamma = 0.0, -0.2, -0.5, -0.7, -0.9$) are depicted in Fig. 7.

For the computed solution with different stretching ratios, Fig. 7 shows that the maximum absolute errors decreased rapidly when either the mesh size decreased or the mesh stretching ratio increased. Such a behavior is what we would normally expect for a good numerical scheme for solving the convection diffusion equations.

Figure 8 further demonstrates the effect of the grid stretching on the accuracy of the computed solution. It shows the solution profiles of Problem 2 with Re $= 100$ at $\xi = 1 - h$ with $h = 1/32$. It can be observed that the difference between the computed solution and the exact solution was reduced quickly as the amount of the grid stretching was increased. With a sufficient grid stretching, the computed solution is graphically indistinguishable from the exact solution.

Our multigrid method did not converge for solving Problem 2 when Re > 1000. This is a typical large Re problem case with a stagnation point in Ω, which will be discussed later. However, in order to show that the fourth-order compact scheme can produce accurate solution for this problem with large Re, we used a preconditioned Krylov subspace method

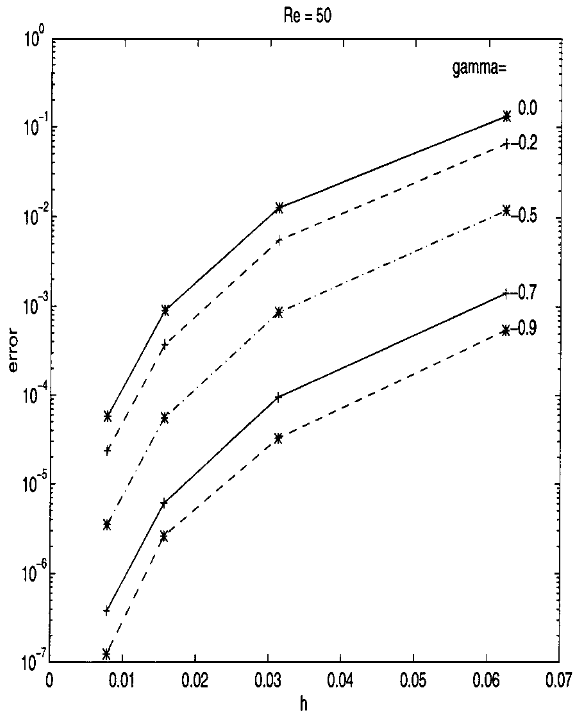

FIG. 7. Comparison of the maximum absolute errors of the computed solutions of Problem 2 (Re $= 50$) with different mesh size h and different stretching ratios γ.

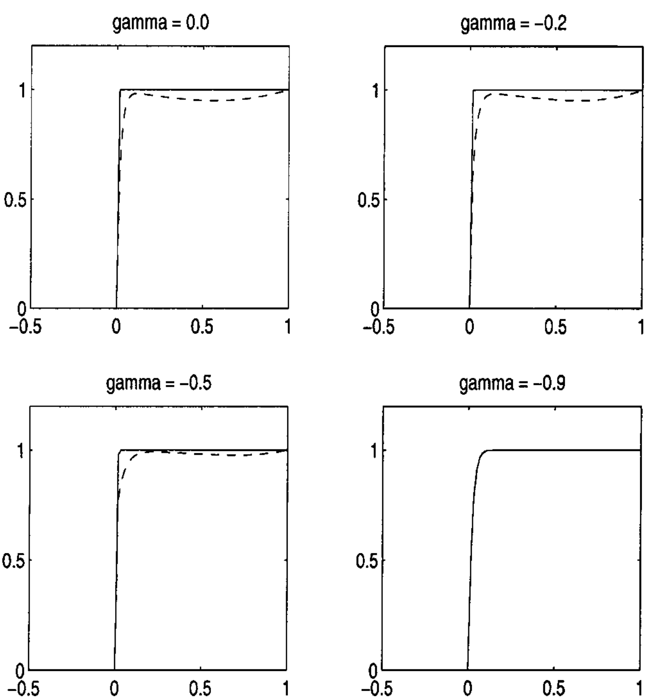

FIG. 8. Solution profiles of Problem 2 (Re $= 100$) with different stretching ratios at $\xi = 1 - h$ with $h = 1/32$. Solid line is the exact solution, dashed line is the computed solution.

TABLE II
Maximum Absolute Errors in the Computed Solution of Problem 2 with Different Convection Coefficient Re, Stretching Ratio γ, and Discretization Parameter h

Re	$\gamma\backslash h$	1/128	1/256	Order
10^0	0.0	7.86 (−12)	4.97 (−13)	3.98
	−0.9	9.09 (−9)	5.68 (−10)	3.99
10^1	0.0	5.25 (−8)	3.28 (−9)	4.00
	−0.9	4.88 (−8)	3.05 (−9)	4.00
10^2	0.0	9.36 (−4)	6.03 (−5)	3.96
	−0.9	3.38 (−7)	2.11 (−8)	4.00
10^3	−0.9	9.64 (−4)	6.22 (−5)	3.95
	−0.99	1.07 (−5)	6.55 (−7)	4.03
10^4	−0.99	8.58 (−4)	5.16 (−5)	4.06
	−0.999	7.72 (−4)	4.38 (−5)	4.14
10^5	−0.999	5.67 (−3)	2.66 (−4)	4.41

to solve the linear system [34]. The computed results are listed in Table II, which show that fourth-order accuracy is achieved for Problem 2 with Re as large as 10^5, provided sufficient grid stretching is performed.

4.3. *Multigrid Convergence*

In this subsection, we demonstrate how the grid stretching affects the convergence of the multigrid method that we used to solve the resulting sparse linear systems. The multigrid iteration was stopped when the finest grid residual in 2-norm was reduced by 10^{10} orders of magnitude. A total of 40 multigrid iterations were allowed in each test run. We used some random vectors as the initial guess. A symbol "–" in a table indicates lack of convergence.

We first tested the multigrid method with an alternating line Gauss–Seidel (ALGS) smoother for solving Problem 1 with $h = 1/64$. The number of multigrid iterations with respect to different diffusion parameter ϵ and different grid stretching ratio γ are listed in Table III.

TABLE III
Number of Multigrid Iterations with ALGS Relaxation for Problem 1 (h = 1/64) with Respect to the Diffusion Parameter ϵ and the Grid Stretching Ratio γ

$\epsilon\backslash\gamma$	1.1	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
10^{-6}	1	1	1	1	1	3	3	4	5
10^{-5}	1	1	1	1	3	2	5	5	4
10^{-4}	1	1	2	3	2	2	5	4	8
10^{-3}	3	6	8	7	10	8	7	11	—
10^{-2}	5	6	6	8	8	8	9	—	—
10^{-1}	6	6	5	6	—	10	13	15	—
10^0	6	8	7	7	13	6	9	13	13

Some comments on the results in Table III are helpful.

• The degree of the grid stretching affects the convergence rates of the multigrid iterations. From left columns to right columns, the magnitude of the stretching ratio increased, and the number of multigrid iterations required to converge increased also. For certain stretching ratios, the multigrid algorithm did not reach convergence, most likely because of some data generated by the stretched grids. One possible explanation is that for certain values of ϵ and γ, the transformed computational domain contains some stagnation points. It can be shown that the convection coefficient of u_ξ in Eq. (8) vanishes at

$$\xi = -\frac{\ln \epsilon (1 - \exp(-Q))}{Q},$$

if this quantity falls into the interval (0, 1). This is exactly the case in Table III with $\epsilon = 0.1$ and $\gamma = 10^4$. With $h = 1/64$, we can see the coefficient of u_ξ vanishes at $\xi \approx 0.2461$.

• The data in the upper left corner of Table III indicate that the multigrid algorithm converged extremely fast when the problem was convection dominated with very small values of ϵ, and when the degree of the grid stretching was not too severe. This phenomenon is not surprising since in this case, the ALGS smoother was a direct solver [25]. What has been missing in literature on this topic is that the computed solution has no accuracy because of the very steep boundary.

• The data in the lower right corner imply that unnecessary grid stretching deteriorated the convergence rate of the multigrid method.

• The data in the upper left corner show that the number of multigrid iterations increased as the stretching ratio increased. However, in these cases, the computed solution obviously has a certain degree of accuracy and the iteration numbers are reasonable.

• The data in the lower left corner represent the cases in which no grid stretching or only a slight grid stretching were needed. The iteration numbers are reasonable. These cases represent the classical multigrid method applied to elliptic problems.

The best known property of geometric multigrid method is probably its grid independent convergence rate when applied to elliptic problems. We also tested the effect of the grid stretching on the convergence dependence with respect to the grid size. Our test was conducted for Problem 1 with $\epsilon = 10^{-3}$. The results are listed in Table IV. We find that grid independent convergence rate was achieved when the grid was not stretched. With a grid stretching, the multigrid convergence rate is dependent on the grid size. However, the very nature of such a dependence is interesting. With a light stretching, the convergence rate was higher when the mesh was refined. With a severe stretching, the convergence rate was deteriorated when the mesh was refined.

TABLE IV
Number of Multigrid Iterations with ALGS Relaxation for Problem 1 ($\epsilon = 10^{-3}$) with Respect to Different Mesh Size *h* and Different Grid Stretching Ratio γ

$h \backslash \gamma$	1.1	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8
1/32	3	4	11	6	7	6	4	6	—
1/64	3	6	8	7	10	8	7	11	—
1/128	3	8	6	11	15	11	9	15	—

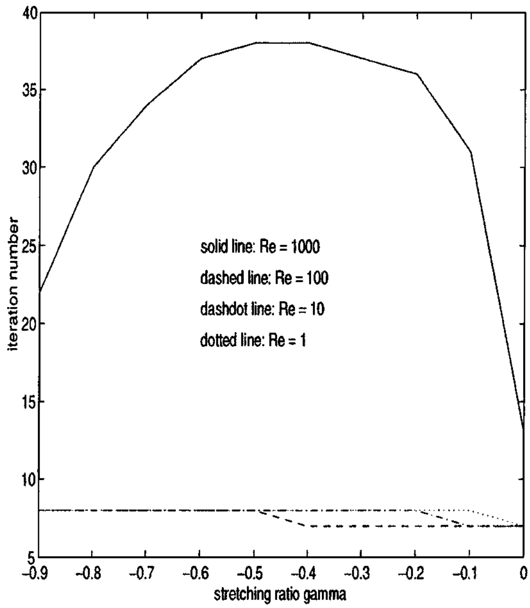

FIG. 9. Number of multigrid iterations for solving Problem 2 with different Reynolds number with respect to the grid stretching ratio γ.

We also tested our multigrid iteration algorithm for solving Problem 2. Since this problem has a stagnation point in its domain, we can only obtain multigrid convergence for Re $\leq 10^3$. For larger Reynolds number cases, special intergrid transfer operators, such as the scaled residual injection technique, are needed to recover reasonable convergence rate for a multigrid method [29, 30, 33, 34]. Such topics are beyond the scope of this paper.

Figure 9 shows the effect of the stretching ratio on the convergence rate of the multigrid iterations for solving Problem 2 with different Reynolds numbers. We note that the multigrid convergence rate was not affected substantially by the effect of the grid stretching when Re ≤ 100. However, the solid line in Fig. 9 indicates that the grid stretching did affect the multigrid convergence substantially when Re $= 1000$. It is interesting to point out that too much and too little grid stretching made the multigrid algorithm converge slightly faster in the case of Re $= 1000$.

5. CONCLUDING REMARKS

We have investigated the use of a multigrid solution technique with a fourth-order nine-point compact finite difference scheme to solve the two-dimensional convection diffusion equation with boundary layers. The boundary layers are first resolved by using a nonuniformly discretized grid so that a few grid points can be placed into the boundary layers. An orthogonal coordinate transformation is then employed to transform the nonuniform grid into a uniform grid, on which the fourth-order compact scheme is applied.

The effect of the stretching ratio on the accuracy of the computed solution is demonstrated by solving two test problems. The numerical results indicate that a nonuniform grid is necessary for solving convection diffusion problems with boundary layers. Without a mesh grading technique, the fourth-order compact scheme is hopeless with respect to a boundary layer, since there is no grid point inside the boundary layer. However, with a mesh grading technique and a suitable coordinate transformation strategy, the high-order accuracy of the computed solution from the fourth-order compact scheme can be recovered. We have shown that the fourth-order accuracy can be obtained for certain highly convection dominated problems with high Reynolds numbers, if a suitable grid stretching is utilized. This result was not previously known. In fact, the computed solution for Problem 1 is shown to be fourth-order accurate at very high Reynolds numbers on stretched grids. This is in contrast with the situation in which a fourth-order compact scheme is used to solve convection diffusion problems without boundary layers on uniform grids and the computed solution is not fourth order accurate at high Reynolds numbers [21, 26, 35].

The multigrid method was shown to be very powerful to solve certain discretized boundary layer problems. For the present two test problems, the transformed equations have variable coefficients, the multigrid method with the alternating line Gauss–Seidel relaxation works just fine. However, we did observe the negative effect of the grid stretching on the convergence rate of the multigrid iterations. We find that, in order to solve a convection diffusion problem with boundary layers, a reasonable degree of grid stretching is necessary.

REFERENCES

1. R. F. Boisvert, Families of high order accurate discretizations of some elliptic problems, *SIAM J. Sci. Statist. Comput.* **2**(3), 268 (1981).
2. A. Brandt, Multi-level adaptive solutions to boundary-value problems, *Math. Comp.* **31**(138), 333 (1977).
3. G. F. Carey, *Computational Grids. Generation, Adaption, and Solution Strategies* (Taylor & Francis, Washington, DC, 1997).
4. P. M. de Zeeuw, Matrix-dependent prolongations and restrictions in a blackbox multigrid solver, *J. Comput. Appl. Math.* **33**, 1 (1990).
5. S. C. R. Dennis and J. D. Hudson, Compact h^4 finite-difference approximations to operators of Navier-Stokes type, *J. Comput. Phys.* **85**, 390 (1989).
6. P. M. Gresho and R. L. Lee, Don't suppress the wiggles—they're telling you something! *Comput. Fluids* **9**, 223 (1981).
7. M. M. Gupta, J. Kouatchou, and J. Zhang, A compact multigrid solver for convection-diffusion equations, *J. Comput. Phys.* **132**, 123 (1997).
8. M. M. Gupta, R. P. Manohar, and J. W. Stephenson, A single cell high order scheme for the convection-diffusion equation with variable coefficients, *Int. J. Numer. Meth. Fluids* **4**, 641 (1984).
9. M. M. Gupta, R. P. Manohar, and J. W. Stephenson, High-order difference schemes for two-dimensional elliptic equations, *Numer. Meth. Partial Differential Eqs.* **1**, 71 (1985).
10. M. M. Gupta and J. Zhang, High accuracy multigrid solution of the 3D convection-diffusion equation, *Appl. Math. Comput.* **113**(2–3), 249 (2000).
11. B. P. Leonard and S. Mokhatari, Beyond first-order upwinding: The ultra-sharp alternative for non-oscillatory steady-state simulation of convection, *Int. J. Numer. Meth. Eng.* **30**, 729 (1990).
12. M. Li, T. Tang, and B. Fornberg, A compact fourth-order finite difference scheme for the steady incompressible Navier–Stokes equations, *Int. J. Numer. Meth. Fluids* **20**, 1137 (1995).
13. R. E. Lynch and J. R. Rice, High accuracy finite difference approximation to solutions of elliptic partial differential equations, *Proceedings of National Academy Sciences U.S.A.* **75**(6), 2541 (1978).
14. B. J. Noye and H. H. Tan, A third-order semi-implicit finite difference method for solving the one-dimensional convection-diffusion equation, *Int. J. Numer. Meth. Eng.* **26**(7), 1615 (1988).
15. A. L. Pardhanani, W. F. Spotz, and G. F. Carey, A stable multigrid strategy for convection-diffusion using high order compact discretization, *Electron. Trans. Numer. Anal.* **6**, 211 (1997).
16. J. Pike, Grid adaptive algorithms for the solution of the Euler equations on irregular grids, *J. Comput. Phys.* **71**, 194 (1987).

17. A. A. Reusken, Multigrid with matrix-dependent transfer operators for convection-diffusion problems, in *Multigrid Method IV, Proceedings of Fourth European Multigrid Conference*, edited by P. W. Hemker and P. Wesseling (Brikhauser Verlag, Basel, 1994), pp. 269–280.

18. P. J. Roache, *Computational Fluid Dynamics* (Hermosa, Albuquerque, NM, 1976).

19. S. G. Rubin and P. K. Khosla, Polynomial interpolation methods for viscous flow calculations, *J. Comput. Phys.* **24**, 217 (1977).

20. J. W. Ruge and K. Stüben, Efficient solution of finite difference and finite element equations, in *Multigrid Methods for Integral and Differential Equations*, edited by D. J. Paddon and H. Holstein (Clarendon Press, Oxford, 1985), pp. 169–212.

21. A. Segal, Aspects of numerical methods for elliptic singular perturbation problems, *SIAM J. Sci. Statist. Comput.* **3**, 327 (1982).

22. W. F. Spotz, *High-Order Compact Finite Difference Schemes for Computational Mechanics*, Ph.D. thesis (University of Texas at Austin, Austin, TX, 1995).

23. W. F. Spotz and G. F. Carey, High-order compact scheme for the steady stream-function vorticity equations, *Int. J. Numer. Methods Eng.* **38**, 3497 (1995).

24. W. F. Spotz and G. F. Carey, Formulation and experiments with high-order compact schemes for nonuniform grids, *Int. J. Numer. Meth. Head & Fluid Flow* **8**(3), 288 (1998).

25. P. Wesseling, *An Introduction to Multigrid Methods* (Wiley, Chichester, England, 1992).

26. I. Yavneh, Analysis of a fourth-order compact scheme for convection-diffusion, *J. Comput. Phys.* **133**, 361 (1997).

27. J. Zhang, Accelerated high accuracy multigrid solution of the convection-diffusion equation with high Reynolds number, *Numer. Meth. Partial Differential Eqs.* **77**(1), 73 (1997).

28. J. Zhang, Minimal residual smoothing in multi-level iterative method, *Appl. Math. Comput.* **84**(1), 1 (1997).

29. J. Zhang, *Multigrid Acceleration Techniques and Applications to the Numerical Solution of Partial Differential Equations*, Ph.D. thesis (The George Washington University, Washington, DC, 1997).

30. J. Zhang, Residual scaling techniques in multigrid, I: Equivalence proof, *Appl. Math. Comput.* **86**(2–3), 283 (1997).

31. J. Zhang, An explicit fourth-order compact finite difference scheme for three dimensional convection-diffusion equation, *Commun. Numer. Meth. Eng.* **14**, 209 (1998).

32. J. Zhang, On convergence and performance of iterative methods with fourth-order compact schemes, *Numer. Meth. Partial Differential Eqs.* **14**, 262 (1998).

33. J. Zhang, Residual scaling techniques in multigrid, II: Practical applications, *Appl. Math. Comput.* **90**(2–3), 229 (1998).

34. J. Zhang, A note on an accelerated high accuracy multigrid solution of the convection-diffusion equation with high Reynolds number, *Numer. Meth. Partial Differential Eqs.* **16**(1), 1 (2000).

35. J. Zhang, Preconditioned iterative methods and finite difference schemes for convection-diffusion, *Appl. Math. Comput.* **109**(1), 11 (2000).